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The homfly and the Kauffman bracket polynomials for the generalized mutant of a link

Taizo Kanenobu [1]

Department of Mathematics, Osaka City University, Sugimoto, Sumiyoshi-Ku, Osaka 558, Japan

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Abstract

Conway's mutation of a link is achieved by flipping a 2-strand tangle. Two mutant links share the same polynomial invariants. Anstee et al. generalized a mutation by flipping a many-string tangle which has rotational symmetry. We give another generalization of mutation: We consider a link constructed with 3-strand tangles $T_1, T_2, \ldots, T_n$ and a $2n$-strand tangle S. Under some conditions, by permuting $T_1, T_2, \ldots, T_n$ or flipping S, the homfly or the Kauffman bracket polynomial do not change.

Keywords: Knot; Link; Mutant; Homfly polynomial; Kauffman bracket polynomial; Skein

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1. Introduction

Let L be a link in S^3. If B is a 3-ball in S^3 whose boundary ∂B meets L transversely in $2n$ points, then $T = B \cap L$ is called a tangle. If $n = 2$, then Conway's mutation of L [3] is achieved by flipping over T or rotating it through angle π. Mutants are formed in three ways; see [14, p.123]. A mutant of L is not necessarily ambient isotopic to L, but has the same polynomial invariants, such as the Alexander, the Conway, the Jones, the homfly (skein, or 2-variable Jones), the Q, and the Kauffman polynomials. When $n \geqslant 3$ and T has an n-fold symmetry, a new link is obtained by dihedrally flipping T. This is called a rotant of L, and was introduced by Anstee, Przytycki, and Rolfsen [1] as a generalization of a mutant.

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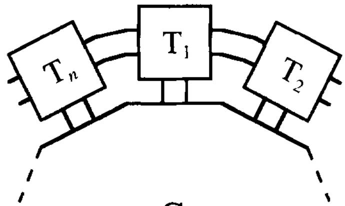


Fig. 1.

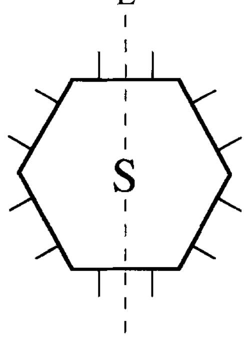


Fig. 2.

They showed that L and its rotant share the same Jones polynomial when $n \leqslant 5$, and the same homfly polynomial when $n \leqslant 4$, and the same Kauffman polynomial when $n = 3$. Jin and Rolfsen [6] produced some examples showing that these restrictions on n cannot be improved.

In this paper, we study the homfly and the Kauffman bracket polynomials of the link (or a link diagram according to the context) $L(S; T_1, T_2, \ldots, T_n)$ of Fig. 1, where S and T_i are tangles. Let S^* be a tangle obtained by dihedrally flipping S; that is, S^* is obtained by rotating S through angle π with axis E as shown in Fig. 2, where $n = 6$. For $n = 2$, $L(S; T_1, T_2)$ and $L(S^*; T_1, T_2)$ are mutants each other, and if $n \geqslant 3$ and $T_1 = T_2 = \cdots = T_n$, then $L(S; T_1, T_2, \ldots, T_n)$ and $L(S^*; T_1, T_2, \ldots, T_n)$ are rotants each other.

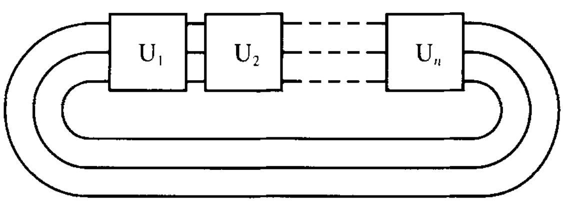


Fig. 3.

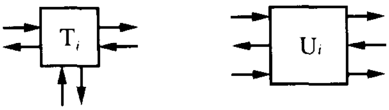


Fig. 4.

Let $L(U_1U_2 \ldots U_n)$ be the link as shown in Fig. 3, which is a special case of the above link. If we treat $L(S;\, T_1,\, T_2, \ldots, T_n)$ and $L(U_1U_2 \ldots U_n)$ as oriented links, we assume that T_i and U_i are oriented as in Fig. 4. In Section 2, we prove that if each U_i satisfies some conditions, then for any permutation σ of the subscripts, the link $L(U_{\sigma(1)}U_{\sigma(2)} \ldots U_{\sigma(n)})$ has the same homfly polynomial as that of $L(U_1U_2 \ldots U_n)$ (Corollary 2.4). This means that the link $L(U_1U_2 \ldots U_n)$ has n-fold symmetry in the homfly polynomial level.

In Section 3, we prove that if each T_i satisfies some conditions, then for any permutation σ of 1, 2, 3, $L(S;\, T_{\sigma(1)},\, T_{\sigma(2)},\, T_{\sigma(3)})$ and $L(S^*;\, T_{\sigma(1)},\, T_{\sigma(2)},\, T_{\sigma(3)})$ have the same homfly polynomials as that of $L(S;\, T_1,\, T_2,\, T_3)$ (Theorem 3.1). This is a generalization of the above mentioned theorem of Anstee et al. in some sense.

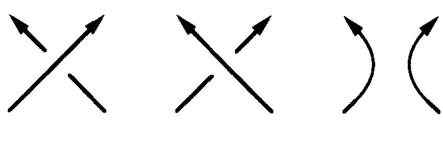
In Sections 4 and 5, we give analogous results for the Kauffman bracket polynomial (Corollary 4.4 and Theorem 5.1). In Section 6, we consider the example of knots $K(p_1,\, p_2, \ldots, p_n)$ introduced in [9] which motivates Theorem 2.3 and Corollary 2.4. In Section 7, we give an example for Theorem 3.1.

We should note that our results are similar in spirit to those obtained by Traczyk [19], Jones [8], and Przytycki [17].

2. The homfly polynomial of $L(U_1U_2 \ldots U_n)$

We use Conway's linear skein theory as described in [14]. A room $\vec{R}$ is a 3-ball with a finite set of points on the boundary $\partial\vec{R}$ each marked either "in" or "out". A tangle is a proper oriented 1-submanifold in R that meets $\partial\vec{R}$ at the designated points in the given direction, and the pre-skein of $\vec{R}$ is the set of isotopy classes relative to $\partial\vec{R}$ of such tangles. The linear skein $\mathscr{L}(\vec{R})$ of $\vec{R}$ is the free module over $\mathbb{Z}[v^{\pm 1},\, z^{\pm 1}]$ generated by the pre-skein of $\vec{R}$ modulo all relations of the form

$$v^{-1}s_+ - vs_- - zs_0 = 0,$$



s_+ s_- s_0

Fig. 5.

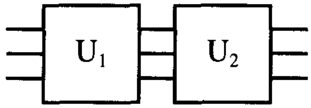


Fig. 6.

where s_+, s_-, and s_0 are generators represented by tangles identical except near one point where they have positive, negative, and vacuous crossings as shown in Fig. 5. For an oriented link L, its homfly polynomial $P_L(v, z) \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ [4,18] in the convention of Morton [15] is given as follows: Suppose that L is in the interior of a 3-ball B, which we consider a room with the empty subset on its boundary. Then $L = P_L(v, z)O$ in $\mathscr{S}(B)$, where O is a trivial knot. Putting $v = 1$, we obtain the Conway polynomial [3], and substituting $(v, z) = (t, t^{1/2} - t^{-1/2})$, we get the Jones polynomial $V_L(t) \in \mathbb{Z}[t^{\pm 1/2}]$ [7].

Let $\vec{R}_3$ be a room having U_i in Fig. 4 as its tangle. By juxtaposing two rooms, two tangles U_1 and U_2 of $\vec{R}_3$ produce a new tangle U_1U_2 as in Fig. 6. This product induces a bilinear map: $\mathscr{S}(\vec{R}_3) \times \mathscr{S}(\vec{R}_3) \to \mathscr{S}(\vec{R}_3)$. As in the introduction, for tangles $U_1, U_2, \ldots, U_n$, we denote a link of Fig. 3 by $L(U_1U_2 \ldots U_n)$, and by $P(U_1U_2 \ldots U_n)$ its homfly polynomial. Then the map $U \mapsto P(U)$ induces a linear map $\mathscr{S}(\vec{R}_3) \to \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$.

Now $\mathscr{S}(\vec{R}_3)$ is generated by the six tangles $\vec{e}_0, \vec{e}_1, \ldots, \vec{e}_5$ as shown in Fig. 7; except for $\vec{e}_5$, they are the same ones as those in [13, Fig. 4]. A multiplication table is given in Table 1, where $\vec{e}_i\vec{e}_j$ appears at the intersection of the ith row and the

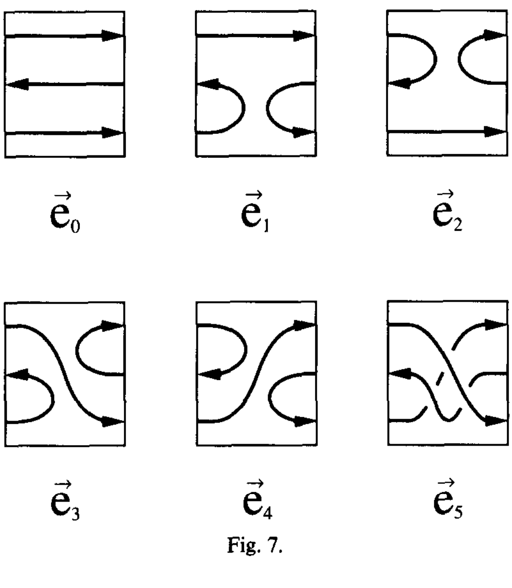


Fig. 7.

Table 1

	$\vec{e}_1$	$\vec{e}_2$	$\vec{e}_3$	$\vec{e}_4$	$\vec{e}_5$
$\vec{e}_1$	$\mu\vec{e}_1$	$\vec{e}_3$	$\mu\vec{e}_3$	$\vec{e}_1$	$\vec{e}_3$
$\vec{e}_2$	$\vec{e}_4$	$\mu\vec{e}_2$	$\vec{e}_2$	$\mu\vec{e}_4$	$\vec{e}_4$
$\vec{e}_3$	$\vec{e}_1$	$\mu\vec{e}_3$	$\vec{e}_3$	$\mu\vec{e}_1$	$\vec{e}_1$
$\vec{e}_4$	$\mu\vec{e}_4$	$\vec{e}_2$	$\mu\vec{e}_2$	$\vec{e}_4$	$\vec{e}_2$
$\vec{e}_5$	$\vec{e}_4$	$\vec{e}_3$	$\vec{e}_2$	$\vec{e}_1$	$v^{-2}\vec{e}_0 - v^{-1}z(\vec{e}_1 + \vec{e}_2 - \vec{e}_5)$

jth column and $\mu = (v^{-1} - v)z^{-1}$ is the homfly polynomial of the trivial 2-component link. If $U = \sum_{i=0}^{5} F^i\vec{e}_i \in \mathscr{S}(\vec{R}_3)$, where $F^i \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$, then

$$\begin{aligned} P(U) &= \sum_{i=0}^{5} F^i P(\vec{e}_i) \\ &= \mu^2 F^0 + \mu(F^1 + F^2) + (F^3 + F^4) + v^{-2}(\mu - vz)F^5. \end{aligned} \tag{1}$$

Similarly, we have

$$\left[P(U),\, P(U\vec{e}_1), \ldots, P(U\vec{e}_5)\right] = [F^0, F^1, \ldots, F^5]M,$$

where

$$M = \begin{bmatrix} \mu^2 & \mu & \mu & 1 & 1 & v^{-2}(\mu - vz) \\ \mu & \mu^2 & 1 & \mu & \mu & 1 \\ \mu & 1 & \mu^2 & \mu & \mu & 1 \\ 1 & \mu & \mu & 1 & \mu^2 & \mu \\ 1 & \mu & \mu & \mu^2 & 1 & \mu \\ v^{-2}(\mu - vz) & 1 & 1 & \mu & \mu & v^{-3}(v\mu^2 - 2\mu v^2 z + \mu z - vz^2) \end{bmatrix}.$$

Then we have

$$H[F^0, F^1, \ldots, F^5] = \left[P(U),\, P(U\vec{e}_1), \ldots, P(U\vec{e}_5)\right]M', \tag{2}$$

where $H = (\mu^2 - 1)((1 + v)^2 - \mu^2 v)((1 - v)^2 + \mu^2 v)$, and $M' = [m_{ij}]$ is the 6×6 symmetric matrix defined by

$$\begin{aligned}
&m_{11} = 1 - \mu^2 - 2v^2 + 3\mu^2 v^2 - \mu^4 v^2 + v^4; \\
&m_{12} = m_{13} = \mu(1 - v^2 + \mu^2 v^2); \\
&m_{14} = m_{15} = m_{23} = -1 + 2v^2 - 2\mu^2 v^2 - v^4; \\
&m_{16} = m_{24} = m_{25} = m_{34} = m_{35} = \mu v^2(-1 + \mu^2 + v^2); \\
&m_{22} = m_{33} = \mu^2(\mu^2 - 2)v^2; \\
&m_{26} = m_{36} = m_{44} = m_{55} = \mu^2 v^2(1 + v^2); \\
&m_{45} = 1 + 2v^2 + 3\mu^2 v^2 - \mu^4 v^2 + v^4 - \mu^2 v^4; \\
&m_{46} = m_{56} = \mu v^2(1 - v^2 + \mu^2 v^2); \\
&m_{66} = \mu^2(\mu^2 - 2)v^4.
\end{aligned}$$

Thus $P(U), P(U\vec{e}_1), \ldots, P(U\vec{e}_5)$ determine $F^0, F^1, \ldots, F^5$ uniquely, so we have

Proposition 2.1. *The six tangles $\vec{e}_0$, $\vec{e}_1, \ldots, \vec{e}_5$ form a free basis for $\mathscr{L}(\vec{R}_3)$.*

This proposition is well known in its full general context (for any tangle or even any product of surface cross interval), cf. [16]. The following is a generalization of [14, Proposition 12].

Proposition 2.2. *Let U_1, $U_2 \in \mathscr{L}(\vec{R}_3)$. Then it holds that*

$$HP(U_1U_2) = \left[P(U_1),\ P(U_1\vec{e}_1), \ldots, P(U_1\vec{e}_5)\right] M' \begin{bmatrix} P(U_2) \\ P(U_2\vec{e}_1) \\ \vdots \\ P(U_2\vec{e}_5) \end{bmatrix}.$$

Proof. Let $U_j = \sum_{i=0}^{5} F_j^i \vec{e}_i$, $j = 1, 2$. Then

$$\begin{aligned} P(U_1U_2) &= P\left(\sum_{i=0}^{5} F_1^i \vec{e}_i \vec{U}_2\right) \\ &= \sum_{i=0}^{5} F_1^i P(\vec{e}_i U_2) \\ &= \left[F_1^0,\ F_1^1, \ldots, F_1^5\right] \begin{bmatrix} P(U_2) \\ P(U_2\vec{e}_1) \\ \vdots \\ P(U_2\vec{e}_5) \end{bmatrix}. \end{aligned}$$

Using (2), we have the desired formula. □

Theorem 2.3. *If each $U_j = \sum_{i=0}^{5} F_j^i \vec{e}_i \in \mathscr{L}(\vec{R}_3)$ satisfies*

$$U_j \vec{e}_2 = G_j \vec{e}_2, \tag{3}$$

where F_j^i, $G_j \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$, then

$$P(U_1U_2 \ldots U_n) = G_1G_2 \ldots G_n + X_1X_2 \ldots X_n + \sum_{k=0}^{n} \phi_k S_{nk}, \tag{4}$$

where

$$X_j = F_j^0 + \mu F_j^1 + F_j^3,$$

ϕ_k is a polynomial defined recursively by

$$\phi_{k+2} = v^{-2}\phi_k + v^{-1}z\phi_{k+1}, \qquad \phi_0 = \mu^2 - 2 \quad \text{and} \quad \phi_1 = v^{-2}(\mu - vz);$$

and

$$S_{nk} = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \frac{F_1^0 F_2^0 \ldots F_n^0}{F_{i_1}^0 F_{i_2}^0 \ldots F_{i_k}^0} F_{i_1}^5 F_{i_2}^5 \ldots F_{i_k}^5.$$

Condition (3) is the algebraic analogy of the geometric concept of the cup-trivial rotor in [1, p.244].

Corollary 2.4. *Let* U_j, $j = 1, 2, \ldots, n$, *be as in Theorem* 2.3. *If* σ *is any permutation of* $1, 2, \ldots, n$, *then*

$$P(U_1 U_2 \ldots U_n) = P(U_{\sigma(1)} U_{\sigma(2)} \ldots U_{\sigma(n)}).$$

Proof of Theorem 2.3. Since

$$\begin{aligned} U_j \vec{e}_2 &= \sum_{i=0}^{5} F_j^i \vec{e}_i \vec{e}_2 \\ &= \left(F_j^0 + \mu F_j^2 + F_j^4\right)\vec{e}_2 + \left(F_j^1 + \mu F_j^3 + F_j^5\right)\vec{e}_3, \end{aligned}$$

by using Proposition 2.1, (3) implies

$$F_j^0 + \mu F_j^2 + F_j^4 = G_j, \tag{5}$$

$$F_j^1 + \mu F_j^3 + F_j^5 = 0. \tag{6}$$

Similarly, we have

$$U_j \vec{e}_1 = X_j \vec{e}_1 + Y_j \vec{e}_4,$$

where $Y_j = F_j^2 + \mu F_j^4 + F_j^5$.

We prove (4) by induction on n. By using (5), (1) yields (4) with $n = 1$. Suppose that (4) is true for $n = N$. Let $U = U_N U_{N+1} = \sum_{i=0}^5 F^i \vec{e}_i$. Then

$$\left.\begin{aligned} F^0 &= F_N^0 F_{N+1}^0 + v^{-2} F_N^5 F_{N+1}^5; \\ F^1 &= F_N^1 F_{N+1}^0 + X_N F_{N+1}^1 + \left(F_N^3 - v^{-1} z F_N^5\right) F_{N+1}^5; \\ F^2 &= F_N^2 F_{N+1}^0 + G_N F_{N+1}^2 + Y_N F_{N+1}^3 + \left(F_N^4 - v^{-1} z F_N^5\right) F_{N+1}^5; \\ F^3 &= F_N^3 F_{N+1}^0 + X_N F_{N+1}^3 + F_N^1 F_{N+1}^5; \\ F^4 &= F_N^4 F_{N+1}^0 + Y_N F_{N+1}^1 + G_N F_{N+1}^4 + F_N^2 F_{N+1}^5; \\ F^5 &= F_N^0 F_{N+1}^5 + F_N^5 F_{N+1}^0 + v^{-1} z F_N^5 F_{N+1}^5. \end{aligned}\right\} \tag{7}$$

Since $U\vec{e}_2 = G_N G_{N+1} \vec{e}_2$, the inductive hypothesis implies

$$P(U_1 \ldots U_{N-1} U) = G_1 \ldots G_{N-1} G_N G_{N+1} + X_1 \ldots X_{N-1} X + \sum_{k=0}^{N} \phi_k S'_{Nk},$$

where

$$X = F^0 + \mu F^1 + F^3,$$

and

$$S'_{Nk} = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant N} \frac{F_1^{0\prime} F_2^{0\prime} \ldots F_N^{0\prime}}{F_{i_1}^{0\prime} F_{i_2}^{0\prime} \ldots F_{i_k}^{0\prime}} F_{i_1}^{5\prime} F_{i_2}^{5\prime} \ldots F_{i_k}^{5\prime}$$

with

$$F_j^{i\prime} = \begin{cases} F_j^i & \text{if } 1 \leqslant j \leqslant N-1; \\ F^i & \text{if } j = N. \end{cases}$$

Thus we have only to show the following:

$$X = X_N X_{N+1}; \tag{8}$$

$$\sum_{k=0}^{N} \phi_k S'_{Nk} = \sum_{k=0}^{N+1} \phi_k S_{N+1,k}. \tag{9}$$

The proof of (8) is easy, so we will prove only (9). If $2 \leqslant k \leqslant N-1$, then

$$\begin{aligned}
S'_{Nk} &= F_N^{0\prime} S_{N-1,k} + F_N^{5\prime} S_{N-1,k-1} \\
&= F^0 S_{N-1,k} + F^5 S_{N-1,k-1} \\
&= \left(F_N^0 F_{N+1}^0 + v^{-2} F_N^5 F_{N+1}^5\right) S_{N-1,k} \\
&\quad + \left(F_N^0 F_{N+1}^5 + F_N^5 F_{N+1}^0 + v^{-1} z F_N^5 F_{N+1}^5\right) S_{N-1,k-1} \\
&= F_N^0 F_{N+1}^0 S_{N-1,k} + \left(F_N^0 F_{N+1}^5 + F_N^5 F_{N+1}^0\right) S_{N-1,k-1} \\
&\quad + \left(v^{-2} S_{N-1,k} + v^{-1} z S_{N-1,k-1}\right) F_N^5 F_{N+1}^5 \\
&= S_{N+1,k} - F_N^5 F_{N+1}^5 S_{N-1,k-2} + \left(v^{-2} S_{N-1,k} + v^{-1} z S_{N-1,k-1}\right) F_N^5 F_{N+1}^5 \\
&= S_{N+1,k} + \left(v^{-2} S_{N-1,k} + v^{-1} z S_{N-1,k-1} - S_{N-1,k-2}\right) F_N^5 F_{N+1}^5.
\end{aligned}$$

For the other cases, we have:

$$\begin{aligned}
S'_{N0} &= F_1^0 F_2^0 \ldots F_{N-1}^0 F^0 \\
&= F_1^0 F_2^0 \ldots F_{N-1}^0 \left(F_N^0 F_{N+1}^0 + v^{-2} F_N^5 F_{N+1}^5\right) \\
&= S_{N+1,0} + v^{-2} F_1^0 F_2^0 \ldots F_{N-1}^0 F_N^5 F_{N+1}^5; \\
S'_{N1} &= F^0 S_{N-1,1} + F_1^0 F_2^0 \ldots F_{N-1}^0 F^5 \\
&= \left(F_N^0 F_{N+1}^0 + v^{-2} F_N^5 F_{N+1}^5\right) S_{N-1,1} \\
&\quad + F_1^0 F_2^0 \ldots F_{N-1}^0 \left(F_N^0 F_{N+1}^5 + F_N^5 F_{N+1}^0 + v^{-1} z F_N^5 F_{N+1}^5\right) \\
&= S_{N+1,1} + \left(v^{-2} S_{N-1,1} + v^{-1} z F_1^0 F_2^0 \ldots F_{N-1}^0\right) F_N^5 F_{N+1}^5; \\
S'_{NN} &= F_1^5 F_2^5 \ldots F_{N-1}^5 F^5 \\
&= F_1^5 F_2^5 \ldots F_{N-1}^5 \left(F_N^0 F_{N+1}^5 + F_N^5 F_{N+1}^0 + v^{-1} z F_N^5 F_{N+1}^5\right) \\
&= F_1^5 F_2^5 \ldots F_{N-1}^5 \left(F_N^0 F_{N+1}^5 + F_N^5 F_{N+1}^0\right) + v^{-1} z S_{N+1,N+1}.
\end{aligned}$$

Using these formulas, we get

$$
\begin{aligned}
\sum_{k=0}^{N} \phi_k S'_{Nk} = {} & \phi_0\big(S_{N+1,0} + v^{-2}F_1^0F_2^0 \dots F_{N-1}^0 F_N^5 F_{N+1}^5\big) \\
& + \phi_1\big\{S_{N+1,1} + \big(v^{-2}S_{N-1,1} + v^{-1}zF_1^0F_2^0 \dots F_{N-1}^0\big)F_N^5F_{N+1}^5\big\} \\
& + \sum_{k=2}^{N-1} \phi_k\big\{S_{N+1,k} \\
& \qquad + \big(v^{-2}S_{N-1,k} + v^{-1}zS_{N-1,k-1} - S_{N-1,k-2}\big)F_N^5F_{N+1}^5\big\} \\
& + \phi_N\big\{F_1^5F_2^5 \dots F_{N-1}^5\big(F_N^0F_{N+1}^5 + F_N^5F_{N+1}^0\big) + v^{-1}zS_{N+1,N+1}\big\} \\
= {} & \sum_{k=0}^{N-1} \phi_k S_{N+1,k} \\
& + \Bigg\{v^{-2}\phi_0F_1^0F_2^0 \dots F_{N-1}^0 + \phi_1\big(v^{-2}S_{N-1,1} + v^{-1}zF_1^0F_2^0 \dots F_{N-1}^0\big) \\
& \qquad + \sum_{k=2}^{N-1} \phi_k\big(v^{-2}S_{N-1,k} + v^{-1}zS_{N-1,k-1} - S_{N-1,k-2}\big)\Bigg\}F_N^5F_{N+1}^5 \\
& + \phi_N\big\{F_1^5F_2^5 \dots F_{N-1}^5\big(F_N^0F_{N+1}^5 + F_N^5F_{N+1}^0\big) + v^{-1}zS_{N+1,N+1}\big\} \\
= {} & \sum_{k=0}^{N-1} \phi_k S_{N+1,k} \\
& + \Bigg\{v^{-2}\phi_1S_{N-1,1} + \phi_2F_1^0F_2^0 \dots F_{N-1}^0 \\
& \qquad + \sum_{k=2}^{N-1} \phi_k\big(v^{-2}S_{N-1,k} + v^{-1}zS_{N-1,k-1} - S_{N-1,k-2}\big)\Bigg\}F_N^5F_{N+1}^5 \\
& + \phi_N\big\{F_1^5F_2^5 \dots F_{N-1}^5\big(F_N^0F_{N+1}^5 + F_N^5F_{N+1}^0\big) + v^{-1}zS_{N+1,N+1}\big\}.
\end{aligned}
$$

The result follows from the following:

$$
\begin{aligned}
& \Bigg\{v^{-2}\phi_1S_{N-1,1} + \phi_2F_1^0F_2^0 \dots F_{N-1}^0 \\
& \quad + \sum_{k=2}^{N-1} \phi_k\big(v^{-2}S_{N-1,k} + v^{-1}zS_{N-1,k-1} - S_{N-1,k-2}\big)\Bigg\}F_N^5F_{N+1}^5 \\
& + v^{-1}z\phi_NS_{N+1,N+1} \\
& \quad = \Bigg(v^{-2}\phi_1S_{N-1,1} + \phi_2F_1^0F_2^0 \dots F_{N-1}^0 + \sum_{k=2}^{N-1} v^{-2}\phi_kS_{N-1,k} \\
& \qquad + \sum_{k=1}^{N-2} v^{-1}z\phi_{k+1}S_{N-1,k} - \sum_{k=0}^{N-3} \phi_{k+2}S_{N-1,k}\Bigg)F_N^5F_{N+1}^5
\end{aligned}
$$

$$+v^{-1}z\phi_N S_{N+1,N+1}$$
$$=\left(\sum_{k=1}^{N-1} v^{-2}\phi_k S_{N-1,k}+\sum_{k=1}^{N-2} v^{-1}z\phi_{k+1}S_{N-1,k}-\sum_{k=1}^{N-3}\phi_{k+2}S_{N-1,k}\right)F_N^5F_{N+1}^5$$
$$+v^{-1}z\phi_N S_{N+1,N+1}$$
$$=\left(\sum_{k=1}^{N-2}\phi_{k+2}S_{N-1,k}+v^{-2}\phi_{N-1}S_{N-1,N-1}-\sum_{k=1}^{N-3}\phi_{k+2}S_{N-1,k}\right)F_N^5F_{N+1}^5$$
$$+v^{-1}z\phi_N S_{N+1,N+1}$$
$$=\left(\phi_N S_{N-1,N-2}+v^{-2}\phi_{N-1}F_1^5F_2^5\dots F_{N-1}^5\right)F_N^5F_{N+1}^5+v^{-1}z\phi_N S_{N+1,N+1}$$
$$=\phi_N S_{N+1,N}-F_1^5F_2^5\dots F_{N-1}^5\left(F_N^0F_{N+1}^5+F_N^5F_{N+1}^0\right)+\phi_{N+1}S_{N+1,N+1}.$$

Thus (9) is true for $n=N+1$, and the proof of the theorem is complete. □

Although Theorem 2.3 yields Corollary 2.4 directly, the above proof is a tedious calculation. Przytycki has kindly informed the author of the following simple proof due to Traczyk and himself. Notice that Corollary 2.4 holds for all satellites for which the link $L(U_1U_2\dots U_n)$ forms the pattern. In fact, a satellite knot on a knot K is considered as $L(U_1U_2\dots U_n\tilde{K})$, where $\tilde{K}$ is the tangle given from the 1-string tangle whose closure is K by replacing the strand by three parallel ones. Then $\tilde{K}\vec{e}_2=P_K(v,z)\vec{e}_2$. This also follows from the next proof.

Proof of Corollary 2.4. By using (5) and (6), (7) yields

$$F^1=\left(F_N^1F_{N+1}^0+F_N^0F_{N+1}^1\right)+\mu F_N^1F_{N+1}^1-\mu F_N^3F_{N+1}^3-v^{-1}zF_N^5F_{N+1}^5;$$
$$F^2=\left(F_N^0F_{N+1}^2+F_N^2F_{N+1}^0\right)+F_N^2F_{N+1}^2-\mu F_N^4F_{N+1}^4-v^{-1}zF_N^5F_{N+1}^5$$
$$+\left(F_N^4Y_{N+1}+Y_NF_{N+1}^3\right);$$
$$F^3=\left(F_N^0F_{N+1}^3+F_N^3F_{N+1}^0\right)-F_N^1F_{N+1}^1+F_N^3F_{N+1}^3;$$
$$F^4=\left(F_N^0F_{N+1}^4+F_N^4F_{N+1}^0\right)-F_N^2F_{N+1}^2+F_N^4F_{N+1}^4+\left(Y_NF_{N+1}^1+F_N^2Y_{N+1}\right).$$

Thus we have

$$[U_N,U_{N+1}]=U_NU_{N+1}-U_{N+1}U_N$$
$$=W_3\vec{e}_2+W_1\vec{e}_4,$$

where

$$W_i=Y_N\left(F_{N+1}^i-F_{N+1}^{i+1}\right)-\left(F_N^i-F_N^{i+1}\right)Y_{N+1}.$$

From (3), we have

$$U_j\vec{e}_4=G_j\vec{e}_4.$$

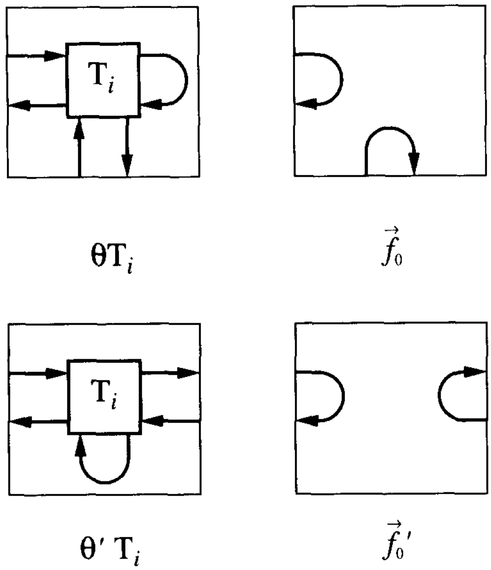


Fig. 8.

Using (3) and this, we have

$$\begin{aligned} P(U_1U_2\dots U_{N-1}[U_N, U_{N+1}]) &= P\left(U_1U_2\dots U_{N-1}\left(W_3\vec{e}_2 + W_1\vec{e}_4\right)\right) \\ &= G_1G_2\dots G_{N-1}P\left(W_3\vec{e}_2 + W_1\vec{e}_4\right) \\ &= G_1G_2\dots G_{N-1}P([U_N, U_{N+1}]) \\ &= 0. \end{aligned}$$

This completes the proof of the corollary. □

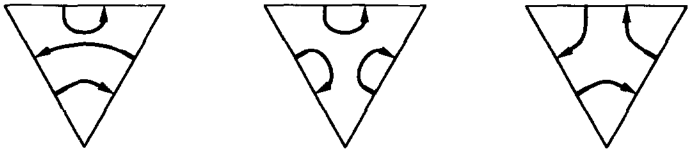
3. The homfly polynomial of $L(S;\, T_1, T_2, T_3)$

Let $L(S;\, T_1, T_2, T_3)$ be the oriented link given in the introduction, and let $P(S;\, T_1, T_2, T_3)$ be its homfly polynomial. Let $\vec{R}'_3$ and $\vec{R}''_3$ be the rooms with tangles S and T_i, respectively. Let θT_i and $\vec{f}_0$ (respectively $\theta' T_i$ and $\vec{f}'_0$) be the tangles of the room $\vec{R}_2$ (respectively $\vec{R}'_2$) as shown in Fig. 8. Suppose that each T_i satisfies the following conditions:

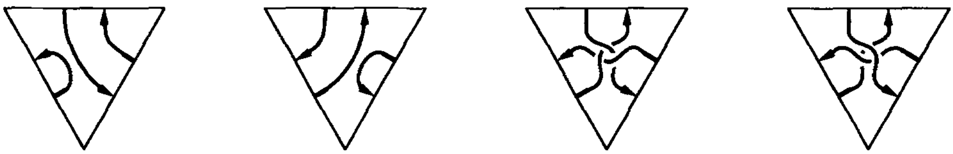
$$\theta T_i = G f_0 \quad \text{in } \mathscr{L}\left(\vec{R}_2\right); \tag{10}$$

$$\theta' T_i = G' f'_0 \quad \text{in } \mathscr{L}\left(\vec{R}'_2\right), \tag{11}$$

where $G_i, G'_i \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$. Then it follows that $G_i = G'_i$, and we have:



$\vec{s}_0$ $\vec{s}_1$ $\vec{s}_2$



$\vec{s}_3$ $\vec{s}_4$ $\vec{s}_5$ $\vec{s}_6$

Fig. 9.

Theorem 3.1. *Let σ be a permutation of* 1, 2, 3. *Then*

$$P(S;\, T_1,\, T_2,\, T_3) = P(S;\, T_{\sigma(1)},\, T_{\sigma(2)},\, T_{\sigma(3)}) = P(S^*;\, T_{\sigma(1)},\, T_{\sigma(2)},\, T_{\sigma(3)}).$$

Proof. By Proposition 2.1, the six tangles $\vec{s}_0$, $\vec{s}_1, \ldots, \vec{s}_4$, and $\vec{s}_6$ in Fig. 9 form a free basis for $\mathscr{L}(\vec{R}'_3)$. Then from the relation $\vec{s}_5 = -v^2z^2\vec{s}_1 + v^3z\vec{s}_3 + v^3z\vec{s}_4 + v^4\vec{s}_6$, the six tangles $\vec{s}_0$, $\vec{s}_1, \ldots, \vec{s}_5$ form a free basis for $\mathscr{L}(\vec{R}'_3)$. So, we may put $S = \sum_{i=0}^{5} F^i\vec{s}_i$, $F^i \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$. Then

$$P(S;\, T_1,\, T_2,\, T_3) = \sum_{i=0}^{5} F^i P(\vec{s}_i;\, T_1,\, T_2,\, T_3).$$

From the conditions, we have

$$P(\vec{s}_i;\, T_1,\, T_2,\, T_3) = \begin{cases} \mu G_1G_2G_3 & \text{if } i = 0, 3, 4; \\ \mu^2 G_1G_2G_3 & \text{if } i = 1. \end{cases}$$

Let T'_i be a tangle of $\vec{R}_3$ as shown in Fig. 10, where all the orientations of T_i are reversed. Then the link $L(\vec{s}_5;\, T_1,\, T_2,\, T_3)$ is isotopic to $L(\vec{T}'_1\vec{T}'_2\vec{T}'_3)$. Therefore

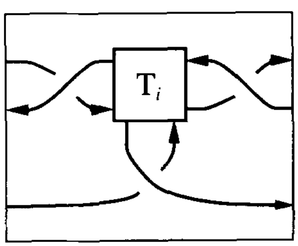


Fig. 10.

Corollary 2.4 implies $P(\vec{s_i}; T_1, T_2, T_3) = P(\vec{s_i}; T_{\sigma(1)}, T_{\sigma(2)}, T_{\sigma(3)})$ for $i = 2, 5$, and so the first equality of the formula is proved.

Since $\vec{s_i^*} = \vec{s_i}$ for $i = 0, 1, 2, 5$ and $\vec{s_3^*} = \vec{s_4}$, we have $P(S; T_1, T_2, T_3) = P(S^*; T_1, T_2, T_3)$, and the second equality follows. This completes the proof. □

4. The Kaufmann bracket polynomial of $L(U_1U_2 \ldots U_n)$

We describe the bracket linear skein theory corresponding to the Kauffman bracket polynomial, which is a simple version of the linear skein theory. The unoriented room and its tangle are just the room and its tangle as defined in Section 2 with their orientations forgotten. We will use the notation R to denote the unoriented room corresponding to the room $\vec{R}$. The regular pre-skein of R is the set of regular isotopy classes relative to ∂R of the tangles. The bracket linear skein $\mathscr{S}^{\text{br}}(R)$ of R is the free module over $\mathbb{Z}[A^{\pm 1}]$ generated by the regular pre-skein of R modulo all relations of the form

$$D \sqcup O = \delta D;$$

$$D_+ = AD_0 + A^{-1}D_\infty,$$

where $\sqcup$ denotes disjoint union, O is a simple closed curve in the diagram, $\delta = -A^2 - A^{-2}$, and D_+, D_0, D_∞ are diagrams identical except near one point where they are as shown in Fig. 11. For a link diagram L, its Kauffman bracket polynomial $\langle L \rangle \in \mathbb{Z}[A^{\pm 1}]$ [10] is given as follows: Suppose that L is in the interior of a 3-ball B, which we consider the room with the empty subset on its boundary. Then $L = \langle L \rangle O$ in $\mathscr{S}^{\text{br}}(B)$. If L is an oriented link diagram and w is its writhe, which is the sum of the signs of the crossing points of D, then the Jones polynomial of L is obtained by putting $A = t^{-1/4}$ in $(-A)^{-3w}\langle L \rangle$.

Now we consider the bracket linear skein theory for the room R_3. In $\mathscr{S}^{\text{br}}(R_3)$, we can define a product as in $\mathscr{S}(\vec{R}_3)$. When a nonzero complex number is substituted for A, $\mathscr{S}^{\text{br}}(R_3)$ is the third Temperley–Lieb algebra; see [12]. For a tangle U of R_3, we define a link diagram $L(U)$ as for $\vec{R}_3$, and we denote by $\langle U \rangle$ its bracket polynomial, which induces a linear map $\mathscr{S}^{\text{br}}(R_3) \to \mathbb{Z}[A^{\pm 1}]$. It is known that $\mathscr{S}^{\text{br}}(R_3)$ is generated by the five tangles $e_0, e_1, \ldots, e_4$, which are

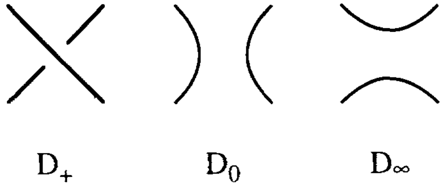


Fig. 11.

$\vec{e}_0, \vec{e}_1, \ldots, \vec{e}_4$ in Fig. 8 with orientations forgotten; see [12, Fig. 7]. If $U = \sum_{i=0}^{4} \alpha^i e_i \in \mathscr{S}^{\mathrm{br}}(R_3)$, where $\alpha^i \in \mathbb{Z}[A^{\pm 1}]$, then we have

$$[\langle U \rangle, \langle Ue_1 \rangle, \ldots, \langle Ue_4 \rangle] = [\alpha^0, \alpha^1, \ldots, \alpha^4] W,$$

where

$$W = \begin{bmatrix} \delta^2 & \delta & \delta & 1 & 1 \\ \delta & \delta^2 & 1 & \delta & \delta \\ \delta & 1 & \delta^2 & \delta & \delta \\ 1 & \delta & \delta & 1 & \delta^2 \\ 1 & \delta & \delta & \delta^2 & 1 \end{bmatrix},$$

and so we obtain

$$h[\alpha^0, \alpha^1, \ldots, \alpha^4] = [\langle U \rangle, \langle Ue_1 \rangle, \ldots, \langle Ue_4 \rangle] W',$$

where $h = (\delta^2 - 2)(\delta^2 - 1)$ and

$$W' = \begin{bmatrix} \delta^2 - 1 & -\delta & -\delta & 1 & 1 \\ -\delta & \delta^2 & 2 & -\delta & -\delta \\ -\delta & 2 & \delta^2 & -\delta & -\delta \\ 1 & -\delta & -\delta & 1 & \delta^2 - 1 \\ 1 & -\delta & -\delta & \delta^2 - 1 & 1 \end{bmatrix}.$$

The following are analogous to Propositions 2.1 and 2.2, Theorem 2.3, and Corollary 2.4, and Proposition 4.1 is again well known [16].

Proposition 4.1. *The five tangles $e_0, e_1, \ldots, e_4$ form a free basis for $\mathscr{S}^{\mathrm{br}}(R_3)$.*

Proposition 4.2. *Let $U_1, U_2 \in \mathscr{S}^{\mathrm{br}}(R_3)$. Then*

$$h\langle U_1 U_2 \rangle = [\langle U_1 \rangle, \langle U_1 e_1 \rangle, \ldots, \langle U_1 e_4 \rangle] W' \begin{bmatrix} \langle U_2 \rangle \\ \langle U_2 e_1 \rangle \\ \vdots \\ \langle U_2 e_4 \rangle \end{bmatrix}.$$

Theorem 4.3. *If each $U_j = \sum_{i=0}^{4} \alpha_j^i e_i \in \mathscr{S}^{\mathrm{br}}(R_3)$ satisfies*

$$U_j e_2 = \beta_j e_2,$$

where $\alpha_j^i, \beta_j \in \mathbb{Z}[A^{\pm 1}]$, then

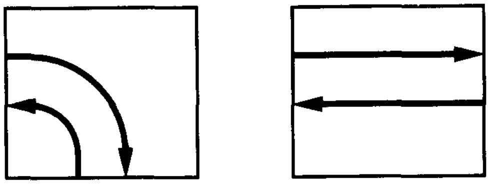
$$\langle U_1 U_2 \ldots U_n \rangle = \beta_1 \beta_2 \ldots \beta_n + x_1 x_2 \ldots x_n + (\delta^2 - 2) \alpha_1^0 \alpha_2^0 \ldots \alpha_n^0,$$

where

$$x_j = \alpha_j^0 + (1 - \delta^2) \alpha_j^3.$$

Corollary 4.4. *Let U_j, $j = 1, 2, \ldots, n$ be as in Theorem 4.3. If σ is any permutation of the subscripts, then*

$$\langle U_1 U_2 \ldots U_n \rangle = \langle U_{\sigma(1)} U_{\sigma(2)} \ldots U_{\sigma(n)} \rangle.$$



$\vec{f}_\infty$ $\vec{f}_\infty{}'$

Fig. 12.

Proof of Theorem 4.3. Since

$$\begin{aligned} U_j e_2 &= \sum_{i=0}^{4} \alpha_j^i e_i e_2 \\ &= \left(\alpha_j^0 + \delta\alpha_j^2 + \alpha_j^4\right)e_2 + \left(\alpha_j^1 + \delta\alpha_j^3\right)e_3, \end{aligned}$$

we obtain

$$\alpha_j^0 + \delta\alpha_j^2 + \alpha_j^4 = \beta_j,$$

$$\alpha_j^1 + \delta\alpha_j^3 = 0.$$

Using these formulas, we can complete the proof by induction on n. □

5. The Kauffman bracket polynomial of $L(S;\, T_1, T_2, \ldots, T_n)$

In this section, we consider the bracket polynomial $\langle S;\, T_1, T_2, \ldots, T_n \rangle$ of the link diagram $L(S;\, T_1, T_2, \ldots, T_n)$ given in Fig. 1. Let R'_n and R''_3 be the rooms having S and T_i as their tangles, respectively. Let f_0, f'_0 be the tangles $\vec{f}_0$, $\vec{f}'_0$ with orientations forgotten, and f_∞, f'_∞ be the tangles given in Fig. 12. Then f_0 and f_∞ (respectively f'_0 and f'_∞) form a free basis for $\mathscr{L}^{\mathrm{br}}(R_2)$ (respectively $\mathscr{L}^{\mathrm{br}}(R'_2)$); cf. [12].

Suppose that each T_j satisfies the following conditions:

$$\theta T_j = \beta_j f_0 \quad \text{in } \mathscr{L}^{\mathrm{br}}(R_2); \tag{12}$$

$$\theta' T_j = \beta'_j f'_0 \text{ in } \mathscr{L}^{\mathrm{br}}(R'_2), \tag{13}$$

where $\beta_j, \beta'_j \in \mathbb{Z}[A^{\pm 1}]$. Then it follows that $\beta_j = \beta'_j$, and we have:

Theorem 5.1. *For any permutation σ of the subscripts,*

$$\langle S;\, T_1, T_2, \ldots, T_n \rangle = \langle S;\, T_{\sigma(1)}, T_{\sigma(2)}, \ldots, T_{\sigma(n)} \rangle.$$

Furthermore, if $n \leqslant 7$, then

$$\langle S;\, T_1, T_2, \ldots, T_n \rangle = \langle S^*;\, T_{\sigma(1)}, T_{\sigma(2)}, \ldots, T_{\sigma(n)} \rangle.$$

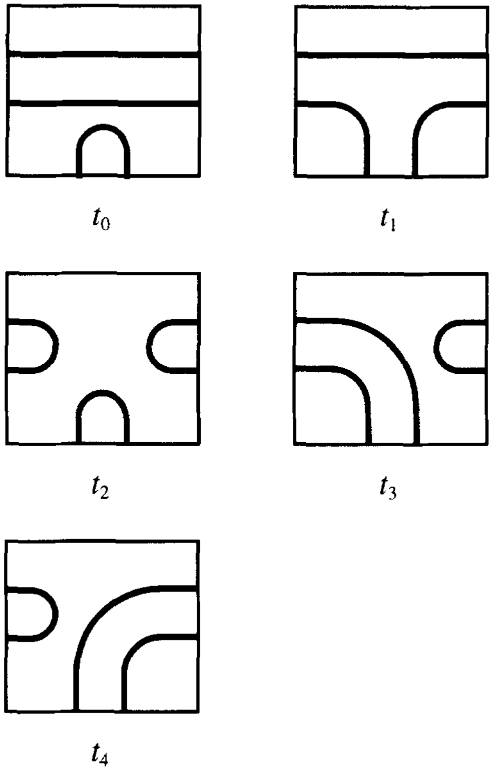


Fig. 13.

Proof. By Proposition 4.1, the five tangles $t_0, t_1, \dots, t_4$ in Fig. 13 form a free basis for $\mathscr{L}(R_3'')$. So, we may put $T_j = \Sigma_{i=0}^4 \alpha_j^i t_i$, where $\alpha_j^i \in \mathbb{Z}[A^{\pm 1}]$. Then

$$\theta T_j = \left(\alpha_j^0 + \delta\alpha_j^2 + \alpha_j^4\right) f_0 + \left(\alpha_j^1 + \delta\alpha_j^3\right) f_\infty;$$

$$\theta' T_j = \left(\delta\alpha_j^2 + \alpha_j^3 + \alpha_j^4\right) f_0' + \left(\delta\alpha_j^0 + \alpha_j^1\right) f_\infty'.$$

Combining these formulas with (12) and (13), we obtain

$$\begin{aligned} \alpha_j^1 &= -\delta\alpha_j^0; \\ \alpha_j^3 &= \alpha_j^0; \\ \alpha_j^4 &= \beta_j - \alpha_j^0 - \delta\alpha_j^2. \end{aligned} \tag{14}$$

Now we abbreviate $\langle S; T_1, T_2, \dots, T_n \rangle$ as $\langle T_1, T_2 \rangle$. From (13), we have

$$\langle t_2, T_2 \rangle = \delta\beta_2 \dots \beta_n \langle \hat{S} \rangle;$$

$$\langle t_4, T_2 \rangle = \beta_2 \dots \beta_n \langle \hat{S} \rangle,$$

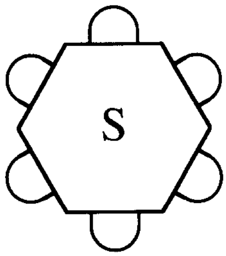


Fig. 14.

where $\hat{S}$ is the link as shown in Fig. 14 (with $n = 6$). And thus we have

$$\begin{aligned}\langle T_1, T_2\rangle &= \left\langle \sum_{i=0}^{4} \alpha_1^i t_i, T_2 \right\rangle \\ &= \sum_{i=0}^{4} \alpha_1^i \langle t_i, T_2\rangle \\ &= \alpha_1^0\langle t_0, T_2\rangle + \alpha_1^1\langle t_1, T_2\rangle + \alpha_1^3\langle t_3, T_2\rangle + \left(\delta\alpha_1^2 + \alpha_1^4\right)\beta_2\dots\beta_n\langle\hat{S}\rangle \\ &= \alpha_1^0\left(\langle t_0, T_2\rangle - \delta\langle t_1, T_2\rangle + \langle t_3, T_2\rangle\right) + \left(\beta_1 - \alpha_1^0\right)\beta_2\dots\beta_n\langle\hat{S}\rangle,\end{aligned}$$

where we use (14). Since

$$\begin{aligned}&\langle t_0, t_2\rangle = \delta\beta_3\dots\beta_n\langle\hat{S}\rangle; \\ &\langle t_0, t_4\rangle = \beta_3\dots\beta_n\langle\hat{S}\rangle; \\ &\delta\langle t_1, t_2\rangle = \langle t_3, t_2\rangle; \\ &\delta\langle t_1, t_4\rangle = \langle t_3, t_4\rangle; \\ &\delta\langle t_3, t_0\rangle = \langle t_3, t_3\rangle,\end{aligned}$$

we have the following:

$$\begin{aligned}&\langle t_0, T_2\rangle - \delta\langle t_1, T_2\rangle + \langle t_3, T_2\rangle \\ &\quad = \sum_{i=0}^{4}\alpha_2^i\langle t_0, t_i\rangle - \delta\sum_{i=0}^{4}\alpha_2^i\langle t_1, t_i\rangle + \sum_{i=0}^{4}\alpha_2^i\langle t_3, t_i\rangle \\ &\quad = \alpha_2^0\langle t_0, t_0\rangle + \alpha_2^1\langle t_0, t_1\rangle + \delta\alpha_2^2\beta_3\dots\beta_n\langle\hat{S}\rangle + \alpha_2^3\langle t_0, t_3\rangle + \alpha_2^4\beta_3\dots\beta_n\langle\hat{S}\rangle \\ &\qquad - \delta\alpha_2^0\langle t_1, t_0\rangle - \delta\alpha_2^1\langle t_1, t_1\rangle - \delta\alpha_2^3\langle t_1, t_3\rangle + 2\alpha_2^0\langle t_3, t_0\rangle + \alpha_2^1\langle t_3, t_1\rangle.\end{aligned}$$

Using (12), this becomes

$$\alpha_2^0\epsilon + \left(\beta_2 - \alpha_2^0\right)\beta_3\dots\beta_n\langle\hat{S}\rangle,$$

where

$$\begin{aligned}\epsilon = {}&\langle t_0, t_0\rangle - \delta\langle t_0, t_1\rangle + \langle t_0, t_3\rangle - \delta\langle t_1, t_0\rangle + \delta^2\langle t_1, t_1\rangle - \delta\langle t_1, t_3\rangle \\ &+ 2\langle t_3, t_0\rangle - \delta\langle t_3, t_1\rangle.\end{aligned}$$

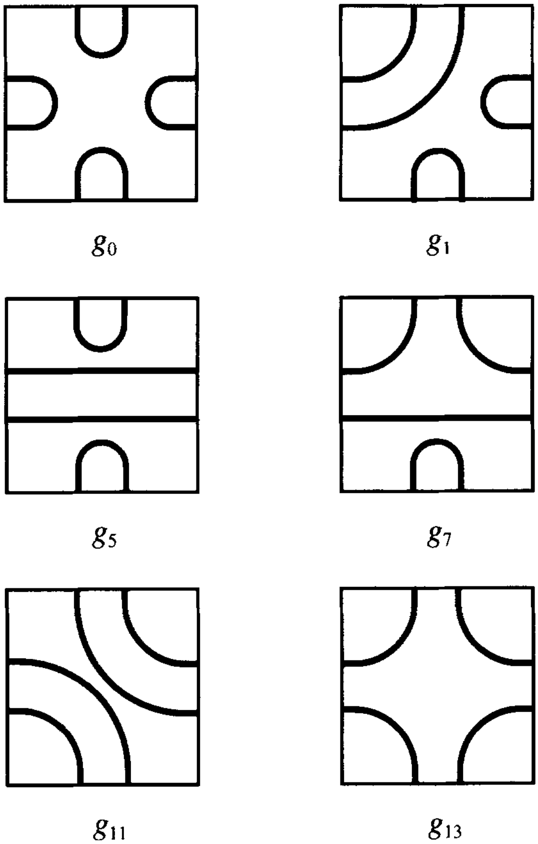


Fig. 15.

Again using (14), we obtain

$$\langle T_1, T_2 \rangle = \alpha_1^0 \alpha_2^0 \epsilon + (\beta_1 \beta_2 - \alpha_1^0 \alpha_2^0) \beta_3 \ldots \beta_n \langle \hat{S} \rangle$$

Thus we have

$$\langle T_1, T_2 \rangle = \langle T_2, T_1 \rangle,$$

and so the proof of the first half is complete.

Next we prove the latter half. It is known that $\mathscr{S}^{\mathrm{br}}(R'_n)$ is generated by all the tangles having neither crossings nor closed components. The number of such generators is the Catalan number $\binom{2n}{n}/(n+1)$. We only consider the case $n = 4$. The generators are $g_0, g_1, \ldots, g_{13}$, where g_0, g_1, g_5, g_7, g_{11}, g_{13} are the tangles as shown in Fig. 15, and the others are given by

$$g_i = \begin{cases} \rho^{i-1} g_1 & \text{if } i = 2, 3, 4; \\ \rho g_5 & \text{if } i = 6; \\ \rho^{i-7} g_7 & \text{if } i = 8, 9, 10; \\ \rho g_{11} & \text{if } i = 12, \end{cases}$$

with ρ the rotation by angle $\pi/2$ around the center of R'_4. Let $S=\sum_{i=0}^{13}\gamma_i g_i$, where $\gamma_i \in \mathbb{Z}[A^{\pm 1}]$. From (13), we have

$$\langle g_i;\, T_1, T_2, T_3, T_4\rangle = \begin{cases} \delta^3\beta & \text{if } i=0;\\ \delta^2\beta & \text{if } 1 \leqslant i \leqslant 6;\\ \delta\beta & \text{if } 7 \leqslant i \leqslant 10, \end{cases}$$

where $\beta=\beta_1\beta_2\beta_3\beta_4$, and thus, we obtain

$$\begin{aligned}\langle S;\, T_1, T_2, T_3, T_4\rangle &= \sum_{i=0}^{13}\gamma_i\langle g_i;\, T_1, T_2, T_3, T_4\rangle\\ &= \beta\left(\delta^3\gamma_0+\delta^2\sum_{i=1}^{6}\gamma_i+\delta\sum_{i=7}^{10}\gamma_i\right)\\ &\quad+\sum_{i=11}^{13}\gamma_i\langle g_i;\, T_1, T_2, T_3, T_4\rangle.\end{aligned}$$

Using the first formula of the theorem, we have

$$\begin{aligned}\langle g_{11}^*;\, T_1, T_2, T_3, T_4\rangle &= \langle g_{12};\, T_1, T_2, T_3, T_4\rangle\\ &= \langle g_{12};\, T_3, T_4, T_1, T_2\rangle\\ &= \langle g_{11};\, T_1, T_2, T_3, T_4\rangle,\end{aligned}$$

and in the same way, we have

$$\langle g_{12}^*;\, T_1, T_2, T_3, T_4\rangle = \langle g_{12};\, T_1, T_2, T_3, T_4\rangle.$$

Since $g_{13}^*=g_{13}$, we obtain

$$\langle S^*;\, T_1, T_2, T_3, T_4\rangle = \langle S;\, T_1, T_2, T_3, T_4\rangle.$$

The other cases can be proved in the same way; cf. [1, Theorem 4.1(a)]. This completes the proof. □

Remark. We have no example of links for which the second formula for $n \geqslant 8$ does not hold.

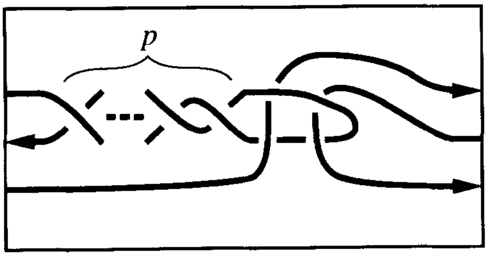


Fig. 16.

6. $K(p_1, p_2, \dots, p_n)$, $n \geqslant 3$, revisited

Let $U(p)$ be the tangle of $\vec{R}_3$ (or R_3) as shown in Fig. 16, where p denotes the number of half twists, right handed if $p > 0$, left if $p < 0$. Note that $U(-p)$ is the mirror image of $U(p-1)$ in $\vec{R}_3$. Then $L(U(p_1)U(p_2)\dots U(p_n))$ is the knot $K(p_1, p_2, \dots, p_n)$ given in [9, Section 5]. Denote by $P(p_1, p_2, \dots, p_n)$, $V(p_1, p_2, \dots, p_n)$, and $\Delta(p_1, p_2, \dots, p_n)$, its homfly, Jones, and Alexander polynomials, respectively. Let $\epsilon_i = 0$ if p_i is even and -1 if p_i is odd. Then it has been shown [9, Proposition 5.3] that

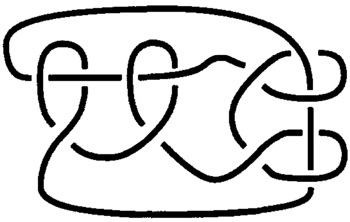
$$P(p_1, p_2, \dots, p_n) = v^{-\sum_{i=1}^n (p_i - \epsilon_i)}\left(P(\epsilon_1, \epsilon_2, \dots, \epsilon_n) - 1\right) + 1;$$

$$V(p_1, p_2, \dots, p_n) = (-t)^{-\sum_{i=1}^n p_i}\left(V(0, 0, \dots, 0) - 1\right) + 1;$$

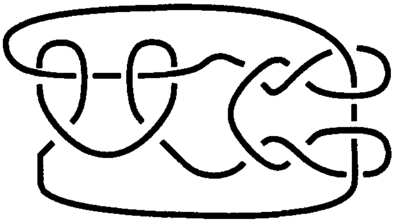
$$\Delta(p_1, p_2, \dots, p_n) = f(t) f(t^{-1}),$$

where $f(t) = (-t)^r - (1-t)^n$, r being the number of 0 in $\epsilon_1, \epsilon_2, \dots, \epsilon_n$. Also it has been shown [9, Proposition 5.2] that if $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i$ and $p_i \equiv q_{\tau(i)} \pmod 2$ for all i, where τ is a cyclic permutation of $1, 2, \dots, n$, then $K(p_1, p_2, \dots, p_n)$ and $K(q_1, q_2, \dots, q_n)$ are skein equivalent: we refer to [14] for the definition of a skein equivalence. Since $U(p)\vec{e}_2 = \vec{e}_2$, from Corollary 2.4, we have

$$P(p_1, p_2, \dots, p_n) = v^{-\sum_{i=1}^n (p_i - \epsilon_i)}\Big(P\big(\underbrace{0, 0, \dots, 0}_{r}, \underbrace{-1, -1, \dots, -1}_{n-r}\big) - 1\Big) + 1.$$



$K(0,-1,0,-1)$



$K(0,0,-1,-1)$

Fig. 17.

In particular, let us consider $K(0, -1, 0, -1)$ and $K(0, 0, -1, -1)$ given in Fig. 17. They have the same homfly polynomial, but have distinct Q polynomials [2,5], which are

$$17 - 32x - 76x^2 + 60x^3 + 192x^4 + 2x^5 - 208x^6 - 102x^7 + 60x^8 + 66x^9 + 20x^{10} + 2x^{11}$$

and

$$17 - 44x^2 - 36x^3 + 88x^4 + 96x^5 - 88x^6 - 128x^7 + 4x^8 + 60x^9 + 28x^{10} + 4x^{11}.$$

Remark and question. The example of the knots $K(0, -1, 0, -1)$ and $K(0, 0, -1, -1)$ led the author to Theorem 2.3 and Corollary 2.4. If two knots are skein equivalent, then they have the same homfly polynomial and have the same signature. Are these two knots skein equivalent? Note that $K(p_1, p_2, \dots, p_n)$ is a ribbon knot, and so its signature is zero. Furthermore, classify $K(p_1, p_2, \dots, p_n)$'s up to ambient isotopy and up to skein equivalence.

Let $n = lm$, where l, m are positive integers. Suppose $p_i = p_j$ for $i \equiv j \bmod m$. Then $K(p_1, p_2, \dots, p_n)$ has period m. There are several criteria for the polynomial invariants of periodic knots. Put $K_1 = K(0, -1, 0, -1, 0, -1)$, $K_2 = K(0, 0, -1, -1, 0, -1)$, and $K_3 = K(0, 0, 0, -1, -1, -1)$. Then K_1 has period 3, and K_2 and K_3 do not seem to have period 3. However, these three knots have the same homfly polynomial, and so the homfly, Jones, Conway, and Alexander polynomials fail to show that K_2 and K_3 do not have period 3. This fact can be proved by using the Kauffman polynomial: Let $F_{ij} \in \mathbb{Z}[a^{\pm 1}]$ be the coefficient of x^j in the Kauffman polynomial of K_i. By [20], if K_i has period 3, then $F_{i0} + (a - a^{-1})F_{i1} \bmod 3$ must be in $\mathbb{Z}[a^{\pm 3}]$. In fact, we have

$$F_{10} + (a - a^{-1})F_{11} = 3a^{-8} + 13a^{-6} + 24a^{-4} + 27a^{-2} + 21 + 3a^2 - 12a^4 - 11a^6 - 3a^8;$$

$$F_{20} + (a - a^{-1})F_{21} = 2a^{-8} + 7a^{-6} + 10a^{-4} + 13a^{-2} + 21 + 17a^2 + 2a^4 - 5a^6 - 2a^8;$$

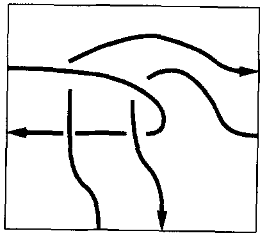
$$F_{30} + (a - a^{-1})F_{31} = a^{-8} + a^{-6} - 4a^{-4} - a^{-2} + 21 + 31a^2 + 16a^4 + a^6 - a^8.$$

7. Example for Theorem 3.1

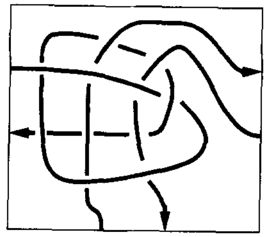
Let S and T_i, $i = 1, 3$, be the tangles of $\vec{R}'_3$ and $\vec{R}''_3$, respectively, as shown in Fig. 18, and T_2 be the mirror image of T_1. Then we have

$$\theta T_1 = \theta T_2 = f_0, \qquad \theta T_3 = G f_0;$$

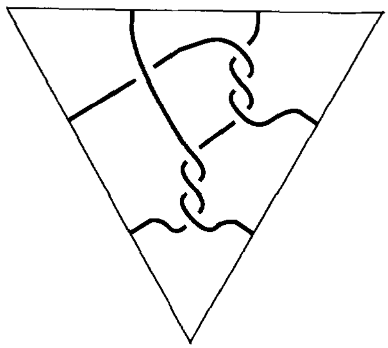
$$\theta' T_1 = \theta' T_2 = f'_0, \qquad \theta' T_3 = G f'_0;$$



T_1



T_3



S

Fig. 18.

where $G = 2v^2 - v^4 + v^2z^2$ is the homfly polynomial of the right-hand trefoil. It follows from Theorem 3.1 that all the following twelve knots share the same homfly polynomial:

$$\big\{L\big(S;\, T_{\sigma(1)},\, T_{\sigma(2)},\, T_{\sigma(3)}\big),$$
$$L\big(S^*;\, T_{\sigma(1)},\, T_{\sigma(2)},\, T_{\sigma(3)}\big) \mid \sigma \text{ is a permutation of } 1, 2, 3\big\}.$$

We can classify these knots by the Q polynomial. Let Q_σ and Q_σ^* be the Q polynomials of $L(S;\, T_{\sigma(1)},\, T_{\sigma(2)},\, T_{\sigma(3)})$ and $L(S^*;\, T_{\sigma(1)},\, T_{\sigma(2)},\, T_{\sigma(3)})$, respectively. Then we have

$$Q_{\text{id}} = -31 + 104x + 72x^2 + \cdots + 2x^{19};$$
$$Q_{(12)} = -31 + 72x + 104x^2 + \cdots + 6x^{19};$$
$$Q_{(13)} = -31 + 104x + 24x^2 + \cdots + 2x^{21};$$
$$Q_{(23)} = -31 + 40x + 136x^2 + \cdots + 4x^{19};$$
$$Q_{(123)} = -31 + 72x + 104x^2 + \cdots + 4x^{18};$$
$$Q_{(132)} = -31 + 40x + 184x^2 + \cdots + 4x^{19};$$

$$Q^*_{\mathrm{id}} = -31 + 104x + 56x^2 + \cdots + 10x^{18};$$
$$Q^*_{(12)} = -31 + 40x + 152x^2 + \cdots + 4x^{19};$$
$$Q^*_{(13)} = -31 + 104x + 40x^2 + \cdots + 2x^{21};$$
$$Q^*_{(23)} = -31 + 72x + 72x^2 + \cdots + 6x^{19};$$
$$Q^*_{(123)} = -31 + 40x + 168x^2 + \cdots + 4x^{18};$$
$$Q^*_{(132)} = -31 + 72x + 136x^2 + \cdots + 6x^{19}.$$

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